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Unsteady heat transfer in the harmonic heating of a dilute suspension of small particles

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Abstract

The governing equation for the temperature response of small particles subjected to diffusive and radiative heat transfer in a homogeneous medium is derived. The method used to derive this integro-differential equation is based on an extension of Duhamel's superposition theorem and it is simpler than the Laplace-transform method traditionally used. This approach is also used to discuss the origin of the history term, which is shown to be a Riemann–Liouville–Weyl half-derivative of the temperature potential between the free-stream and the particle surface. This observation is used to derive the scaling of the unsteady and the quasi-steady contributions for harmonic perturbations of the background temperature field. We identify a scaling number S_R, which is a normalized dimensionless frequency, that measures the importance of the history term as compared to the quasi-steady diffusion and radiation contributions. The relevance of the scaling analysis for turbulent flows is discussed.

1. Introduction

There are many engineering applications where the unsteady heat and/or mass transfer to/from a small particle plays an important role. These applications include solid and liquid fueled flames, particle dryers, industrial separators, and many others systems where either high-frequency temperature variations of the surrounding field occur or the particle is suddenly injected into a temperature field that differs from the particle initial temperature. In these situations, an additional heat transfer contribution caused by the evolving temperature profile around the particle may become relevant to the correct determination of the particle temperature behavior. This work is concerned with the relative scaling of the unsteady and the quasi-steady conduction terms, and the determination of conditions where the unsteady (or *history*) term is of relevance.

Michaelides and Feng [1] derived the energy equation for a particle moving in a non-uniform temperature field in the limit of infinitesimal Péclet ($Pe = Re_p Pr$) and Biot ($Bi = ha/\kappa_p$) numbers.[1] The starting point of their derivation was the realization that the solution for the heat transfer problem due to a step change in temperature presents a term involving the error function. This fact motivates the study of the unsteady conduction term and to relate this term to the *Basset* or *history* drag term appearing in the particle equation of motion for unsteady Stokes flows.

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[1] Given that the *Pe* number condition is imposed, the *Bi* number condition is reduced to $\kappa_p \gg \kappa_m$.

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Nomenclature

a	radius of particle
c	specific heat capacity
C_R	dimensionless radiation coefficient
k_{am}	total absorption coefficient
Q	heat transfer rate
q	heat flux in radial direction
$\mathbf{q}$	heat flux
r	radial coordinate measured from the center of the particle
T	temperature
$\langle T \rangle$	mean temperature level $T_m(0)$
t	time

Greek symbols

α_m	thermal diffusivity of the medium
κ	thermal conductivity
λ	$= \rho_m c_m / \rho_p c_p$
η	Kolmogorov length scale
θ	dimensionless temperature potential, $= (T_p - T_m)/\langle T \rangle$
σ	Stefan–Boltzmann constant
τ	transmissivity or dummy variable of integration
Ω	angular frequency of harmonic temperature field

Subscripts

m	related to the medium
p	related to the particles
r	at position r

Superscripts

V	volumetric
$\oplus$	related to unit temperature jump

Diacritics

$\hat{}$, $*$	dimensionless
$\tilde{}$	related to the medium in the absence of particles
$\bar{}$	due to the presence of the particles

The derivation in [1] closely followed the method used by Maxey and Riley [2], who derived the equation of motion for a small particle moving in a non-uniform velocity field. Their method consisted of Laplace-transformation of the Unsteady Diffusion Equation (UDE) for the near-field formulation, where all the convective terms are neglected a priori. The *history* drag and conduction terms appear from the inversion of the term with a fractional exponent in the resulting Laplace-space algebraic equations. The integral form of the *history* terms appears as a consequence of the application of the Convolution Theorem in the inversion procedure.

The method used by Maxey and Riley [2] and Michaelides and Feng [1] is standard in the literature for reducing the UDE, a partial differential equation, to a linear integro-differential equation [3–6]. This method requires that most of the algebraic derivation be performed in Laplace-space. For this reason, the integral-transform methods often obscure the origin of individual terms and their relationship with the original partial differential equation (PDE). We present here an alternative derivation of the energy equation that shows that the *history* term is a consequence of the fractional (or fractal) nature of the UDE in a semi-infinite medium. We will show that not only the unsteady conduction term is represented by a fractional derivative in time, but also that any unsteady diffusion process occurring in a semi-infinite medium is characterized by a fractional diffusional flux. This characteristic of the UDE in a semi-infinite domain is used to derive the relative scaling of the unsteady and quasi-steady conduction terms.

Section 2 presents an alternate derivation of the *history* term that does not employ integral transformations, but instead employs the Duhamel's Superposition Theorem. The relevance of this method is that there is no need to perform algebraic operations in transformed space, since the entire derivation is performed in physical space. In the same section, it is shown that the kernel of the *history* term matches exactly the definition of a Riemann–Liouville–Weyl half-derivative. The identification of the fractional nature of the history term is important not only to find a general solution for the case of a uniform but time-dependent background temperature field [7], but also to establish the existence of a fractional scale in time for any diffusion process occurring in a semi-infinite domain. This observation also has important consequences for the study of anomalous diffusion processes [8].

Section 2 also presents a direct method for solving the UDE in semi-infinite domains. The method presented here is based on the use of fractional derivative operators. The relevance of Fractional Calculus methods in the analytical formulation of diffusive [7,9,10] and viscoelastic damping [11–13] problems is that a concise and generic formulation that often leads to the solution of the governing equations is generated.

The Fractional Calculus methods also allow the direct scaling of individual terms in the equations, without resorting to transformed-space algebra.

Section 3 addresses the relative scaling of the *history* and quasi-steady conduction terms for a harmonically oscillating background temperature field. This particular background temperature field is of special interest, because the history term contribution is represented by the simplest functional form that does not asymptote to zero for long times. The analysis in Section 3 is confined to a purely diffusive heat transfer problem. Quasi-steady, linear radiation effects are included in the formulation of the heat transfer problem in Section 4, as suggested in [9]. Finally, in Section 5, the equation originally derived in [1] for a non-uniform temperature field is re-derived without resorting to Laplace-transformation of the UDE.

2. The origin of the *history* term

The objective of this section is to derive a governing equation for the heat transfer from a small sphere in an infinite medium. The small, rigid, highly conductive sphere is suspended in a medium that is characterized by a uniform but time-dependent background temperature field. The non-uniformity of the spherically symmetric temperature field is only due to the presence of the particle. A non-uniform background temperature field is considered in Section 5. In the linear limit of $Pe \rightarrow 0$, the total temperature field is composed of two separate parts:

$$T_{\mathrm{m}}(r, t) = \tilde{T}_{\mathrm{m}}(t) + \bar{T}_{\mathrm{m}}(r, t), \tag{1}$$

where the tilde stands for the unperturbed or background temperature field and the over bar stands for the perturbed temperature field of the medium m. The perturbed temperature field is the temperature field that results from the presence of the particle.

The partial differential equation that governs the diffusive heat transfer in the medium m with a uniform but time-dependent volumetric heat source is

$$\frac{\partial T_{\mathrm{m}}}{\partial t} = \alpha_{\mathrm{m}} \nabla^2 T_{\mathrm{m}} + \frac{\tilde{q}^{\mathrm{V}}}{\rho_{\mathrm{m}} c_{\mathrm{m}}}. \tag{2}$$

The boundary and initial conditions to describe the transient heat transfer in the medium that is initially in thermal equilibrium with the particle p of radius a are

$$T_{\mathrm{m}}(r, 0) = \tilde{T}_{\mathrm{m}}(0), \tag{3}$$

$$\bar{T}_{\mathrm{m}}(r, 0) = 0, \tag{4}$$

$$\bar{T}_{\mathrm{m}}(\infty, t) = 0, \tag{5}$$

$$T_{\mathrm{m}}(a, t) = T_{\mathrm{p}}(t), \tag{6}$$

$$\left(\frac{\rho_{\mathrm{p}} c_{\mathrm{p}} a}{3}\right) \frac{\mathrm{d} T_{\mathrm{p}}(t)}{\mathrm{d} t} = q^{\mathrm{in}} = \bar{q}^{\mathrm{in}} + \tilde{q}^{\mathrm{in}} = \kappa_{\mathrm{m}} \frac{\partial \bar{T}_{\mathrm{m}}(r, t)}{\partial r}\bigg|_{r=a} + \tilde{q}^{\mathrm{in}}. \tag{7}$$

The particle temperature equation (7) suggests the division of the problem into two distinct contributions, one from the perturbed field and other from the unperturbed field. Eq. (2) is thus rewritten as

$$\frac{\partial \bar{T}_{\mathrm{m}}}{\partial t} + \frac{\partial \tilde{T}_{\mathrm{m}}}{\partial t} = \alpha_{\mathrm{m}} \nabla^2 \bar{T}_{\mathrm{m}} + \frac{\tilde{q}^{\mathrm{V}}}{\rho_{\mathrm{m}} c_{\mathrm{m}}} \tag{8}$$

The contribution from the unperturbed problem is given by equating the second terms on both sides of Eq. (8). In order to determine the perturbed heat flux input due to the presence of the particle, the associated problem defined by the first terms on both sides of Eq. (8) is considered

$$\frac{\partial \bar{T}_{\mathrm{m}}}{\partial t} = \alpha_{\mathrm{m}} \nabla^2 \bar{T}_{\mathrm{m}}, \tag{9}$$

with

$$\bar{T}_{\mathrm{m}}(\infty, t) = 0, \tag{10}$$

$$\bar{T}_{\mathrm{m}}\left(a, t \geq 0^+\right) = T_{\mathrm{m}}(a, t) - \tilde{T}_{\mathrm{m}}(t) = T_{\mathrm{p}}(t) - \tilde{T}_{\mathrm{m}}(t) = \Delta T_{\mathrm{m}}(t), \tag{11}$$

$$\bar{T}_{\mathrm{m}}(r, 0) = 0. \tag{12}$$

This problem can be solved for a generic $\Delta T_{\mathrm{m}}(t)$ by use of Duhamel's superposition integral if the solution for a unit-temperature jump between the surface temperature and the far field is known. Let the solution of the problem for $\Delta T_{\mathrm{m}}(t) = 1$ be $\bar{T}_{\mathrm{m}}^{\oplus}(r, t)$. Duhamel's Superposition Theorem then gives the following solution for a generic $\Delta T_{\mathrm{m}}(t)$:

$$\bar{T}_{\mathrm{m}}(r, t) = \int_0^t \bar{T}^{\oplus}(r, t - \tau) \frac{\mathrm{d}\bar{T}(a, \tau)}{\mathrm{d}\tau} \mathrm{d}\tau. \tag{13}$$

The perturbed problem is thus reduced to the determination of the temperature profile that results from a unit-temperature jump at the surface. Defining the variables $\Theta(r, t) = r\bar{T}_{\mathrm{m}}(r, t)/a\bar{T}_{\mathrm{m}}(a, t)$, $x = r/a$, and $\tau = \alpha_{\mathrm{m}} t/a^2$, the associate problem for the unit-temperature jump is reduced to

$$\frac{\partial \Theta(x,\tau)}{\partial \tau} = \frac{\partial^2 \Theta(x,\tau)}{\partial x^2}, \tag{14}$$

$$\Theta(x,0) = 0, \quad \Theta\left(1, \tau \geq 0^+\right) = 1, \quad \Theta(\infty, \tau) = 0. \tag{15}$$

The solution of problem (Eqs. (14) and (15)) is well-known, and gives the following expressions for the temperature distribution and heat transfer in terms of the original variables

$$\bar{T}^{\oplus}(r,t) = \frac{a}{r}\mathrm{erfc}\left(\frac{r-a}{\sqrt{4\alpha_m t}}\right), \tag{16}$$

$$\frac{\partial \bar{T}^{\oplus}(r,t)}{\partial r} = -\frac{a}{r^2}\mathrm{erfc}\left(\frac{r-a}{\sqrt{4\alpha_m t}}\right) - \frac{2a}{r\sqrt{4\pi\alpha_m t}}\exp\left[-\frac{(r-a)^2}{4\alpha_m t}\right]. \tag{17}$$

At the surface of the particle, the heat flux is given by

$$\left.\frac{\partial \bar{T}^{\oplus}(r,t)}{\partial r}\right|_{r=a} = -\frac{1}{a} - \frac{1}{\sqrt{\pi\alpha_m t}}. \tag{18}$$

Eq. (13) is then differentiated with respect to r and evaluated at $r = a$ yielding

$$\left.\frac{\partial \bar{T}_m(r,t)}{\partial r}\right|_{r=a} = -\frac{\bar{T}_m(a,t)}{a} - \int_0^t \left(\frac{\mathrm{d}\bar{T}_m(a,t)}{\mathrm{d}\tau}\right)\frac{1}{\sqrt{\pi\alpha_m(t-\tau)}}\,\mathrm{d}\tau. \tag{19}$$

Eqs. (6)–(8) and (19) combined give an equation for the temperature of a particle subjected to a time-dependent, uniform temperature field:

$$\left(\frac{\rho_p c_p a}{3}\right)\frac{\mathrm{d}T_p}{\mathrm{d}t} + \frac{\kappa_m\left(T_p - \tilde{T}_m\right)}{a} + \frac{\kappa_m}{\sqrt{\pi\alpha_m}}\int_0^t \frac{\mathrm{d}\left(T_p - \tilde{T}_m\right)/\mathrm{d}\tau}{\sqrt{(t-\tau)}}\,\mathrm{d}\tau = \left(\frac{\rho_m c_m a}{3}\right)\frac{\mathrm{d}\tilde{T}_m}{\mathrm{d}t}. \tag{20}$$

The remarkable feature of Eq. (20) is that it relates the temperature of the particle to the temperature of the medium if the particle was not present. In dimensionless terms, Eq. (20) is written as

$$\frac{\mathrm{d}\theta}{\mathrm{d}\hat{t}} = (\lambda - 1)\frac{\mathrm{d}\tilde{T}_m^*}{\mathrm{d}\hat{t}} - 3\theta - 3\sqrt{\frac{\lambda}{\pi}}\int_0^{\hat{t}}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\frac{1}{\sqrt{\hat{t}-\tau}}\,\mathrm{d}\tau. \tag{21}$$

In Eq. (21), $\hat{t}$ is the dimensionless time $\kappa_m t/\rho_p c_p a^2$, λ is the heat capacity ratio $\rho_m c_m/\rho_p c_p$, and θ is the dimensionless temperature potential $[T_p(\hat{t}) - \tilde{T}_m(\hat{t})]/\langle T\rangle$, where $\langle T\rangle$ is the characteristic mean temperature level between the particle and the medium, which can be fixed as $\tilde{T}_m(0)$ for small particles. The quantity $\tilde{T}_m^*$ is the *dimensionless*, *unperturbed*, time-dependent temperature of the medium $[\tilde{T}_m(t)/\tilde{T}_m(0)]$.

Eqs. (20) and (21) are valid in the limit of small Péclet and Biot numbers and for uniform background temperature fields. The small Péclet number restriction exists because the convective terms were neglected a priori in the present analysis. The small Biot number restriction exists because the temperature of the particle is assumed to be uniform throughout its volume.

Eq. (21) is used in the next section to determine the relative scaling of the unsteady and steady contributions to the heat transfer rate on a dilute cloud of particles in a time-dependent background field. This equation is the heat transfer analog to *Tchen's first equation* of motion [6]

$$\left(1 + \frac{\alpha}{2}\right)\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\hat{t}} = (\alpha - 1)\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\hat{t}} - \mathbf{w} - \sqrt{\frac{9\alpha}{2\pi}}\int_0^{\hat{t}}\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\sqrt{\hat{t}-\sigma}} + (1-\alpha)\frac{\tau_p \mathbf{g}}{U_0}, \tag{22}$$

where $\mathbf{w}$ is the relative velocity $(\mathbf{v} - \mathbf{u})$, α is the fluid-to-particle density ratio, $\mathbf{g}$ is gravity acceleration vector, τ_p is a particle characteristic time given by $2\rho_p a^2/9\mu$, $\hat{t}$ is the dimensionless time, $\mathbf{v}$ and $\mathbf{u}$ are the dimensionless particle and fluid velocities, respectively. Time, velocity, and length are non-dimensionalized by the flow characteristics τ_p, U_0, and L, respectively. Eq. (22) has been solved analytically in a recent paper [9]. The third term on the right-hand side of Eq. (21) is analogous to the *Basset* or *history* term in the particle momentum equation (the integral term in Eq. (22)) and, therefore, is denominated the *history* term in this work.

Note that Eq. (22) has slightly different coefficients than Eq. (21). The $\alpha/2$ term appearing on the left-hand side of Eq. (22) is a virtual mass coefficient that arises from the non-spherical pressure distribution around the particle in the Stokes equation. Since we assumed uniform temperature distribution for the background field, the heat transfer problem here is spherically symmetric and this term has no parallel in Eq. (21). The last term on the right-hand side of Eq. (22) is a combination of the hydrostatic pressure distribution (buoyancy) and the weight of the particle. This term also presents no parallel in the heat transfer formulation.

It is important to note the functional form of the kernel of the *history* (integral) term in Eqs. (21) and (22). The kernel, which appears from the direct application of Duhamel's Superposition Theorem, has exactly the same form of the Riemann–Liouville–Weyl half-derivative [14]:

$$\frac{\mathrm{d}^{1/2}(\hat{t})}{\mathrm{d}\hat{t}^{1/2}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\hat{t}} \frac{\mathrm{d}f(\sigma)}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\sqrt{\hat{t}-\sigma}} = \frac{1}{\sqrt{\pi}}\int_{0}^{\hat{t}} \frac{\mathrm{d}f(\sigma)}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\sqrt{\hat{t}-\sigma}} + \frac{f(0)}{\sqrt{\pi\hat{t}}}. \quad (23)$$

The fact that the history term is a half-derivative of the temperature or velocity difference between the particle surface value and the condition in the far field is of direct interest for the determination of the solution of the governing equations. This is because the two integro-differential equations that govern the motion and heat transfer of a particle can be readily transformed into second-order ordinary differential equations by application of suitable fractional-operators. The determination of these linear operators and their use are discussed in the next paragraphs.

The following discussion illustrates the natural appearance of the half-derivative in the solution of the diffusion equation in a semi-infinite medium. Again consider again Eq. (14) governing the perturbed temperature field. This equation can be written as

$$\left(\frac{\partial^{1/2}}{\partial\tau^{1/2}} \mp \frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}\Theta(x,\tau)}{\partial\tau^{1/2}} \pm \frac{\partial\Theta(x,\tau)}{\partial x}\right) = Z^{\mp}\left(Z^{\pm}(\Theta)\right) = 0. \quad (24)$$

Upon inspection, we see that by considering the inner bracket $Z^{+}(\Theta) = 0$, we not only satisfy the governing equation (14) but also the associated initial, boundary conditions, and the particle heat balance (15). On the other hand, the conjugate equation $Z^{-}(\Theta) = 0$ leads to unbounded temperature values and thus must be discarded. Since the problem is linear and the solution unique, solving the problem $Z^{+}(\Theta) = 0$ is equivalent to solving Eq. (14) and its associated limiting conditions.

It is of particular interest that the heat flux is given directly by the half-derivative of the temperature potential in the planar configuration. Eq. (21) is readily obtained from $Z^{+}(\Theta) = 0$ when $\bar{T}_{\mathrm{m}}$, r and $\hat{t}$ are recovered from Θ, x and τ. In fact, the classical result of heat flux varying inversely with the square root of time for a sudden jump in temperature is obtained explicitly from the last term on the right-hand side of Eq. (23). Note that any unsteady diffusion problem in a semi-infinite medium that can be reduced to the form of Eqs. (14)–(17) necessarily presents a fractional scale in time. This fundamental result has important implications for the assessment of individual terms in the equations of motion and heat transfer as the scaling analysis of the next section shows.

3. The scaling of the unsteady and steady conduction terms

In this section, we examine the scaling of the unsteady and steady conduction terms in the heat transfer equation for a particle that is subjected to a temperature field that changes sinusoidally with time. After the initial transient caused by the initial condition of thermal equilibrium, the particle temperature engages in a behavior that can also be very closely approximated by a sinusoidal wave. The initial condition $\theta(0) = 0$ does not reflect the periodic motion attained for long times thus the initial transient behavior is expected. For scaling purposes, it is considered here that the temperature potential θ can also be approximated by a sinusoidal function with the same frequency of the background fluid velocity but with amplitude 'b' and a phase difference. In the next section, the exact analytical solution of the problem is presented and the quality of this assumption is assessed. The history and the quasi-steady conduction terms are thus given by

$$C_{\mathrm{H}} = 3\sqrt{\lambda}\frac{\mathrm{d}^{1/2}\theta}{\mathrm{d}\hat{t}^{1/2}} \cong 3\sqrt{\lambda}\frac{\mathrm{d}^{1/2}\left[\sin(\omega\hat{t})\right]}{\mathrm{d}\hat{t}^{1/2}} = 3b\sqrt{\omega\lambda}\frac{\mathrm{d}^{1/2}\left[\sin(\omega\hat{t})\right]}{\left[\mathrm{d}(\omega\hat{t})\right]^{1/2}} \cong O\left(3b\sqrt{\omega\lambda}\right), \quad (25)$$

$$C_{\mathrm{S}} = 3\theta \cong 3b\sin(\omega\hat{t}) \cong O(3b). \quad (26)$$

Note that for the history term, the generalized chain rule for fractional derivatives is used. The chain rule for a generic differ-integral operation is given by [10]

$$\frac{\mathrm{d}^{n}f\left[g(x)\right]}{\mathrm{d}x^{n}} = \frac{f\left[g(x)\right]}{x^{n}\Gamma(1-n)} + \sum_{j=1}^{\infty}\binom{n}{j}\frac{x^{\,j-n}}{\Gamma(j-n+1)}j! \times \sum_{m=1}^{j} f^{\,m}\sum\prod_{k=1}^{j}\frac{1}{P_k!}\left[\frac{f^{\,k}}{k!}\right]^{P_k}, \quad (27)$$

where the last summation extends over all combination of non-negative integer values of P_k such that

$$\sum_{k=1}^{n} kP_k = n, \quad (28)$$

and

$$\sum_{k=1}^{n} P_k = m. \quad (29)$$

For $g(x) = \omega x$, the generalized chain rule gives simply

$$\frac{d^n f(\omega x)}{dx^n} = \omega^n \frac{d^n f(\omega x)}{[d(\omega x)]^n}. \tag{30}$$

The relative scaling of the two dimensionless contributions for the case of a harmonic perturbation is thus

$$\langle C_H : C_S \rangle = \langle (\omega\lambda)^{1/2} : 1 \rangle. \tag{31}$$

The scaling relation (31) shows that when $\lambda\omega = 1$, the amplitude of the history conduction term is equal to the amplitude of the steady conduction term. When the value of the product $\alpha\omega$ is much smaller than 1, the quasi-steady conduction term dominates.

Note that the product $\lambda\omega$ does not depend on the heat capacity of the particle. The dimensionless number that governs the scaling of the conduction terms is

$$S = \sqrt{\lambda\omega} = a\sqrt{\frac{\Omega}{\alpha_m}}, \tag{32}$$

where Ω is the dimensional forcing frequency and α_m is the thermal diffusivity of the surrounding medium.

4. Inclusion of radiation effects and exact solution of the equation

Reference [7] describes the formulation of the unsteady heat transfer problem in a dilute suspension of small particles including linearized radiation effects. The diffusion process is assumed to occur in the vicinity of the particles that are many diameters apart from each other. Radiation effects include the exchange between the free-stream surrounding medium and the cloud of particles. Since the diffusion problem is linear, and the radiation problem can be linearized when considering small particles and relatively small radiation absorption coefficients, the radiation effect is superimposed to the diffusion heat transfer problem. Inclusion of the radiation requires the inclusion of an extra heat flux contribution, which, after non-dimensionalization, becomes

$$\hat{Q}_{\text{rad, in}} = \frac{-12a\varepsilon_m\varepsilon_p\sigma\langle T\rangle^3\theta}{\kappa_m[1-\tau_m(1-\varepsilon_p)]}, \tag{33}$$

where for a gray medium $\tau_m = 1 - \varepsilon_m$, and for a non-gray medium, both τ_m and ε_m are total internal properties. The diffusion–radiation equation describing the transient temperature potential for the particles is then

$$\frac{d\theta}{d\hat{t}} + 3\sqrt{\frac{\lambda}{\pi}}\int_0^{\hat{t}}\left(\frac{d\theta}{d\tau}\right)\frac{1}{\sqrt{\hat{t}-\tau}}\,d\tau + 3(1 + C_R)\theta$$
$$= (\lambda - 1)\frac{d\tilde{T}^*_m}{d\hat{t}}. \tag{34}$$

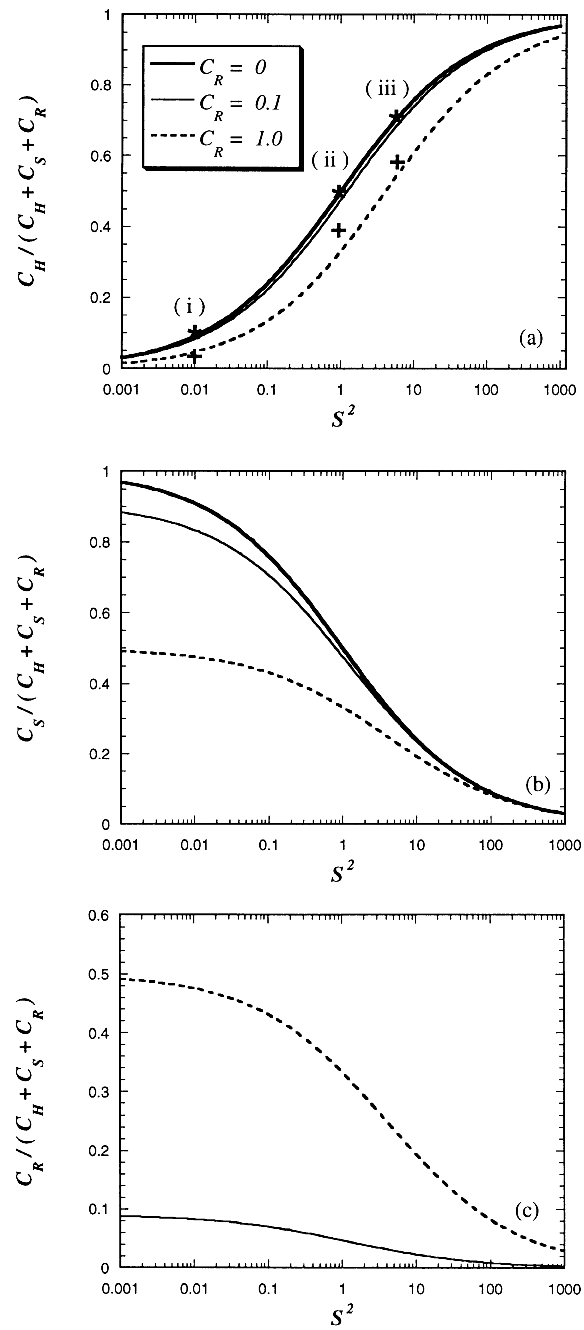


Fig. 1. Relative contribution of the (a) history, (b) quasi-steady conduction, and (c) radiation heat transfer fluxes for different values of C_R. The lines correspond to the scaling analysis of Section 3, and the discrete points correspond to time-averaged values of the full solution for the first 1.5 periods. The discrete values were calculated using: (i) $\omega = 1$, $\lambda = 0.01$, $C_R = 0$ and 1; (ii) $\omega = 10$, $\lambda = 0.1$, $C_R = 0$ and 1; and (iii) $\omega = 100$, $\lambda = 0.1$, $C_R = 0$ and 1.

The dimensionless radiation coefficient C_R is defined as [7]

$$C_R = \frac{4a\varepsilon_m\varepsilon_p\sigma\langle T\rangle^3}{\kappa_m\left[1-\tau_m\left(1-\varepsilon_p\right)\right]}. \tag{35}$$

This coefficient is a measure of the contribution of the radiation mode to the temperature behavior of the particles and has the meaning of a *radiation Nusselt* number. Radiation effects can be neglected if $C_R \ll 1$. However, in many engineering applications, the order of magnitude of C_R maybe ≥ 1, implying that radiation can contribute dominantly to the quasi-steady conduction term in Eq. (34). The restriction on the absorption coefficient for which the approximations made in the radiation modeling are valid can be evaluated in terms of the C_R coefficient. In terms of C_R, k_{am} must satisfy $k_{am} \ll 4\lambda/C_R a$ for the quasi-steady radiation formulation to be valid [7].

Fig. 1 shows the scaling analysis of Section 3 in graphic form. The various curves correspond to the individual contributions (quasi-steady conduction, *history* conduction and quasi-steady radiation) for three different values of the radiation coefficient C_R. Fig. 1(a) shows the *history* term contribution predicted by the scaling analysis. Figs. 1(b) and (c) represent the quasi-steady conduction and the radiation contributions, respectively. Note that for the case of C_R equal to 1, the radiation and the quasi-steady conduction contributions are coincidental. The relative contributions are plotted as functions of the square of the scaling number S. Because of the square root decay of the history term with S, the *history* term contribution is relevant even for values of S^2 of the order of 0.01.

We can also define a new scaling number that includes quasi-steady, linearized radiation effects:

$$S_R = \frac{\sqrt{\lambda\omega}}{(1+C_R)} = \frac{a}{(1+C_R)}\sqrt{\frac{\Omega}{\alpha_m}}. \tag{36}$$

Clearly, the importance of the history term increases linearly with increasing particle radius for negligible radiation. For values of C_R much greater than 1, the influence of the history term becomes independent of the radius of the particle. However, this condition is not likely to be observed, since the thermal conductivity and diffusivity of gases increase with temperature. Very large values of C_R are not likely to occur in real systems.

We now turn to the solution of Eq. (34), which can be obtained analytically following the method outlined in [7]. In order to proceed, we define the following conjugate linear operators $\Psi^\pm$ comprised of three terms, one of them containing a Riemann–Liouville–Weyl half-derivative:

$$\Psi^\pm = \frac{d}{d\hat{t}} \pm H\sqrt{\pi}\frac{d^{1/2}}{d^{1/2}\hat{t}} + D. \tag{37}$$

The left-hand side of Eq. (34) is then simply $\Psi^+[\theta(\hat{t})]$. The initial step in the solution procedure consists of applying the conjugate fractional-differential operator Ψ^- to Eq. (34). The objective of this procedure is to stretch the half-derivative associated with the *history* term in that equation. Application of Ψ^- to Eq. (34)

$$\Psi^-\left\{\Psi^+\left[\theta(\hat{t})\right]\right\} = U\Psi^-\left\{\frac{d\tilde{T}_m^*}{d\hat{t}}\right\} \tag{38}$$

results in

$$\begin{aligned}
&\frac{d^2\theta}{d\hat{t}^2} + (2D - H^2\pi)\frac{d\theta}{d\hat{t}} + D^2\theta \\
&\quad = -U\frac{d^2\tilde{T}_m^*}{d\hat{t}^2} - DU\frac{d\tilde{T}_m^*}{d\hat{t}} \\
&\qquad + HU\left(\frac{1}{\sqrt{\hat{t}}}\frac{d\tilde{T}_m^*}{d\hat{t}}\bigg|_{\hat{t}=0} + \int_0^{\hat{t}}\frac{d^2\tilde{T}_m^*}{d\tau^2}\frac{d\tau}{\sqrt{\hat{t}-\tau}}\right).
\end{aligned} \tag{39}$$

Eq. (39) can now be solved exactly. In order to solve Eq. (39), the nature of the solution has to be analyzed. There is a critical value of λ for which the functional form of the solution changes character. This value is $\lambda_c = 4(1+C_R)/3$, and corresponds to the change in sign of the discriminant $\Delta = H^2\pi(H^2\pi - 4D)$ of the characteristic equation associated with Eq. (39). Three possible cases exist: $\lambda > \lambda_c$, $\lambda = \lambda_c$, and $\lambda < \lambda_c$. The first two cases are of limited practical interest, since they imply a suspension of particles with infinite conductivity (the temperature of the particles is assumed to be uniform throughout their volumes) but with density smaller than the density of the surrounding medium, since the value of the specific heat capacities of solids, liquids and gases are of the same order of magnitude. Because of this implication, the cases related to values of λ greater than or equal to the critical value will not be considered here.

The general solution of Eq. (39) for $\lambda < \lambda_c$ is found, through variation of parameters, to be [7]:

$$\theta(\hat{t}) = \frac{e^{-\alpha\hat{t}}}{\beta}\left\{\begin{array}{l} +U\dfrac{d\tilde{T}_m}{d\hat{t}}\bigg|_{\hat{t}=0}\sin(\beta\hat{t}) \\ -\sin(\beta\hat{t})\displaystyle\int_0^{\hat{t}} e^{\alpha\tau}\cos(\beta\tau)\,\mathrm{rhs}(\tau)\,d\tau \\ +\cos(\beta\hat{t})\displaystyle\int_0^{\hat{t}} e^{\alpha\tau}\sin(\beta\tau)\,\mathrm{rhs}(\tau)\,d\tau \end{array}\right\}, \tag{40}$$

where $\mathrm{rhs}(\hat{t})$ is

$$\mathrm{rhs}(\hat{t}) = HU\left(\int_0^{\hat{t}} \frac{\mathrm{d}^2\tilde{T}^*_\mathrm{m}}{\mathrm{d}\tau^2}\frac{\mathrm{d}\tau}{\sqrt{\hat{t}-\tau}} + \frac{1}{\sqrt{\hat{t}}}\frac{\mathrm{d}\tilde{T}^*_\mathrm{m}}{\mathrm{d}\hat{t}}\bigg|_{\hat{t}=0}\right) - U\frac{\mathrm{d}^2\tilde{T}^*_\mathrm{m}}{\mathrm{d}\hat{t}^2} - DU\frac{\mathrm{d}\tilde{T}^*_\mathrm{m}}{\mathrm{d}\hat{t}}, \quad (41)$$

and the coefficients α and β are defined as

$$\alpha = D - H^2\pi/2, \quad (42)$$

$$\beta = \sqrt{|\Delta|}/2. \quad (43)$$

In order to obtain Eq. (40), the following initial conditions were used:

$$\theta(0) = 0, \quad (44)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\hat{t}}\bigg|_{\hat{t}=0} = U\frac{\mathrm{d}\tilde{T}^*_\mathrm{m}}{\mathrm{d}\hat{t}}\bigg|_{\hat{t}=0}. \quad (45)$$

The first initial condition is due to the assumption of initial thermal equilibrium between the particles and the surrounding medium. The second condition is derived directly from Eq. (34). Note that the initial condition (44) can be relaxed by consideration of the Heavyside solution for the associate problem (Eqs. (14) and (15)).

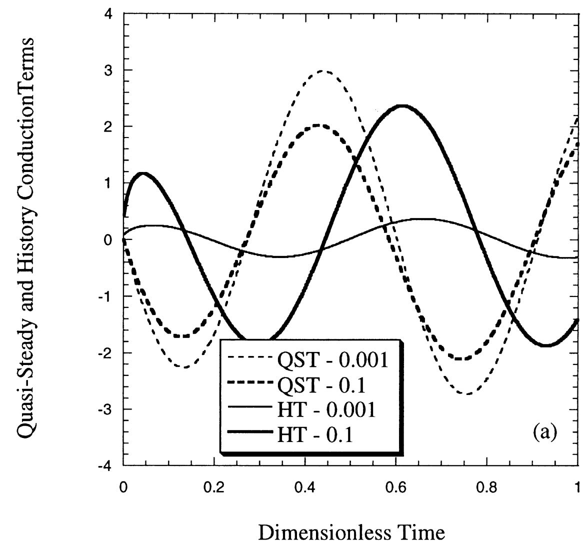
Eq. (40) is thus the general solution for the dimensionless temperature potential. By adding to it the value of the unperturbed temperature of the medium $\tilde{T}^*_\mathrm{m}(\hat{t})$, the dimensionless temperature of the particle is found. For the case of harmonic heating, and to be able to linearize the radiation term, the dimensionless background temperature is given by $\tilde{T}^*_\mathrm{m}(\hat{t}) = 1 + \xi\sin(\omega\hat{t})$, where $\xi \ll 1$. The term $\mathrm{rhs}(\hat{t})$ in this case is

$$\mathrm{rhs_H}(\hat{t}) = \xi\omega U\left[\begin{array}{l} +\omega\sin(\omega\hat{t}) - D\cos(\omega\hat{t}) \\ +\dfrac{H}{\sqrt{\hat{t}}} - \omega H\displaystyle\int_0^{\hat{t}}\frac{\sin(\omega\tau)\,\mathrm{d}\tau}{\sqrt{\hat{t}-\tau}}\end{array}\right]. \quad (46)$$

The dimensionless temperature potential is then given by

$$\theta_\mathrm{H}(\hat{t}) = \frac{\mathrm{e}^{-\alpha\hat{t}}}{\beta}\left\{\begin{array}{l} +U\xi\omega\sin(\beta\hat{t}) \\ -\sin(\beta\hat{t})\displaystyle\int_0^{\hat{t}}\mathrm{e}^{\alpha\tau}\cos(\beta\tau)\,\mathrm{rhs_H}(\tau)\,\mathrm{d}\tau \\ +\cos(\beta\hat{t})\displaystyle\int_0^{\hat{t}}\mathrm{e}^{\alpha\tau}\sin(\beta\tau)\,\mathrm{rhs_H}(\tau)\,\mathrm{d}\tau\end{array}\right\}. \quad (47)$$

Solution (47) is used to validate the scaling analysis of Section 3. The discrete points in Fig. 1(a) represent specific cases where the amplitude of the heat fluxes were averaged over the first 1.5 periods. The results from the exact solution validate the approximate scale analysis even in the presence of the initial transient for cases where phase differences between the heat fluxes are not too accentuated. The characteristic time chosen for non-dimensionalization of the equation $\rho_\mathrm{p}c_\mathrm{p}a^2/\kappa_\mathrm{m}$ is roughly the decay time for the transient caused by the initial condition $\theta(0) = 0$, thus the periodic be-



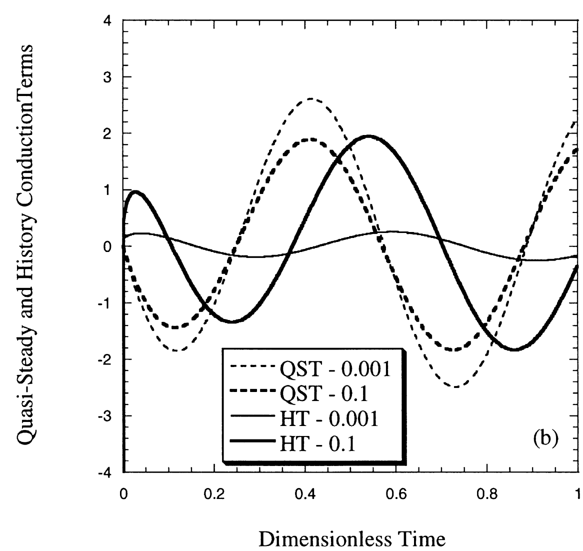


Fig. 2. History and quasi-steady heat fluxes calculated from the analytical solution for $\omega = 10$, and $\alpha = 0.001$ and 0.1. (a) $C_\mathrm{R} = 0$; (b) $C_\mathrm{R} = 1$.

havior is expected to be attained during the first period. Fig. 2 shows the behavior of the history and quasi-steady diffusion heat fluxes for the conditions indicated.

5. Energy equation for a generic temperature field

Consider now a small particle moving through a non-uniform temperature field. Assuming a small *Bi* number for the process, the uniform but time-dependent temperature of the particle is given by

$$m_p c_p \frac{dT_p}{dt} = -\oint_{A_{sp}} \mathbf{q}\cdot\boldsymbol{n}\, dA = -\oint_{A_{sp}} \tilde{\mathbf{q}}\cdot\boldsymbol{n}\, dA - \oint_{A_{sp}} \bar{\mathbf{q}}\cdot\boldsymbol{n}\, dA, \tag{48}$$

where $\mathbf{n}$ is the unit vector perpendicular to the surface of the particle A_{sp} and $\mathbf{q}$ is the total heat flux at the interface between the particle and the fluid. As in the previous sections, the total heat flux can be divided into two parts, the perturbed and the unperturbed heat flux contributions. The unperturbed temperature satisfies the energy equation for the continuous phase [1]

$$-\oint_{A_{sp}} \tilde{\mathbf{q}}\cdot\mathbf{n}\, dA = \rho_m c_m \frac{4\pi a^3}{3}\left(\frac{\partial T_m}{\partial t} + u_i \frac{\partial \tilde{T}_m}{\partial x_i}\right) = \rho_m c_m \frac{4\pi a^3}{3} \frac{D\tilde{T}_m}{Dt}\bigg|_{\mathbf{x}(t)}, \tag{49}$$

where the subscript m indicates properties of the background medium and D/Dt is a substantial derivative following a fluid particle at the position of the rigid particle $\mathbf{x}(t)$.

Note that in Ref. [1], the term containing the substantial derivative was identified as an analog to the *virtual mass* term in the equation of motion derived by Maxey and Riley [2]. This association is not correct since the term described in Eq. (49) is the heat transfer analog to the so-called *pressure term* in the equation of motion. The terminology "*pressure term*" is due to the fact that, for inviscid flows, the acceleration terms are compensated by the pressure distribution around the particle only. The term in the right-hand side of Eq. (49) is more accurately named the *unperturbed Lagrangian transient term*. In the heat transfer formulation, there is no virtual mass term since there is no term equivalent to the pressure term. Furthermore, the heat transfer from the surface of the particle is assumed to be spherically symmetric for infinitesimal Péclet numbers.

In order to determine the contribution from the perturbed temperature field, the unperturbed fluid is expanded as a quadratic function of the coordinate system moving with the particle [1,2,15]. Defining the relative positioning vector as $\mathbf{z} = \mathbf{X} - \mathbf{x}(t)$, where $\mathbf{X}$ is the Eulerian reference frame fixed in time, the unperturbed temperature field is expanded in a McLaurin series

$$\tilde{T}_m(\mathbf{z}, t) \cong \tilde{T}_m(\mathbf{x}, t) + z_i \frac{\partial \tilde{T}_m}{\partial x_i}\bigg|_{\mathbf{z}\to 0} + \frac{1}{2} z_i z_j \frac{\partial^2 \tilde{T}_m}{\partial x_i\, \partial x_j}\bigg|_{\mathbf{z}\to 0} + \cdots \tag{50}$$

The perturbed temperature field is then taken as

$$\bar{T}_m(\mathbf{z}, t) = T_p(t) - \tilde{T}_m(\mathbf{z}, t), \tag{51}$$

which gives

$$\bar{T}_m(\mathbf{z}, t) = T_p(t) - \tilde{T}_m(\mathbf{x}, t) - z_i \frac{\partial \tilde{T}_m}{\partial x_i}\bigg|_{\mathbf{z}\to 0} - \frac{1}{2} z_i z_j \frac{\partial^2 \tilde{T}_m}{\partial x_i\, \partial x_j}\bigg|_{\mathbf{z}\to 0}. \tag{52}$$

In order to determine the perturbed heat flux $\bar{\mathbf{q}}$, it helps noticing that since the surface of the particle is assumed spherical, the third term on the right-hand side of Eq. (52) does not yield a net contribution. This is also true for the non-diagonal terms in the fourth term of the same equation [1]. Through direct application of the inner bracket defined in Eq. (24), $Z^+(\Theta) = 0$, or through application of Duhamel's Superposition Theorem, the heat flux from the perturbed field is given by

$$\bar{\mathbf{q}} = -\kappa_m \frac{\bar{T}_m(\mathbf{z}, t)}{a} - \kappa_m \int_0^t \left(\frac{d\bar{T}_m(\mathbf{z}, t)}{d\tau}\right) \frac{1}{\sqrt{\pi\alpha_m (t-\tau)}}\, d\tau, \tag{53}$$

where d/dt now represents a Lagrangian derivative following the particle. The temperature of the particle in a non-uniform temperature field is then governed by

$$m_p c_p \frac{dT_p}{dt} = \rho_m c_m \frac{4\pi a^3}{3} \frac{D\tilde{T}_m}{Dt}\bigg|_{\mathbf{x}(t)} - \oint_{A_{sp}} \bar{\mathbf{q}}\cdot\mathbf{n}\, dA. \tag{54}$$

Integration of the heat flux (54) over the surface of the particle gives

$$\oint_{A_{\rm sp}} \bar{\mathbf{q}} \cdot \mathbf{n}\, \mathrm{d}A = \frac{\kappa_{\rm m} A_{\rm sp}}{a} \left\{ T_{\rm p}(t) - \tilde{T}_{\rm m}(\mathbf{x}, t) - \frac{a^2}{6} \nabla^2 \tilde{T}_{\rm m}|_{\mathbf{z}\to 0} \right\} + \kappa_{\rm m} A_{\rm sp} \int_0^t \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ T_{\rm p}(t) - \tilde{T}_{\rm m}(\mathbf{x}, t) - \frac{a^2}{6}\nabla^2 \tilde{T}_{\rm m}|_{\mathbf{z}\to 0}\right\}}{\sqrt{\pi \alpha_{\rm m}(t-\tau)}}\, \mathrm{d}\tau. \tag{55}$$

Combination of Eqs. (54) and (55) yields the energy equation for a particle moving in a non-uniform temperature field in the limit of infinitesimal Pe and Bi numbers:

$$m_{\rm p} c_{\rm p} \frac{\mathrm{d}T_{\rm p}}{\mathrm{d}t} = \rho_{\rm m} c_{\rm m} \frac{4\pi a^3}{3} \frac{\mathrm{D}\tilde{T}_{\rm m}}{\mathrm{D}t}\bigg|_{\mathbf{x}(t)} - \frac{\kappa_{\rm m} A_{\rm sp}}{a} \left\{ T_{\rm p}(t) - \tilde{T}_{\rm m}(\mathbf{x}, t) - \frac{a^2}{6}\nabla^2 \tilde{T}_{\rm m}|_{\mathbf{z}\to 0} \right\} - \kappa_{\rm m} A_{\rm sp} \int_0^t \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ T_{\rm p}(t) - \tilde{T}_{\rm m}(\mathbf{x}, t)\right\}}{\sqrt{\pi\alpha_{\rm m}(t-\tau)}}\, \mathrm{d}\tau + \kappa_{\rm m} A_{\rm sp} \int_0^t \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ \frac{a^2}{6}\nabla^2 \tilde{T}_{\rm m}|_{\mathbf{z}\to 0}\right\}}{\sqrt{\pi\alpha_{\rm m}(t-\tau)}}\, \mathrm{d}\tau. \tag{56}$$

Eq. (56) is the energy equation first derived by Michaelides and Feng [1]. The derivation of Eq. (56) presented above shows explicitly that the Faxén corrections (the terms in the Laplacian of the background temperature field) are contributions originated from the perturbed field through consideration of a McLaurin series expansion on $\mathbf{z}$ for the unperturbed field. Because of this, the Faxén corrections must be included in every term associated with the temperature potential between the particle and the unperturbed field. This conclusion is in contrast with the heuristic argument used by Tchen [6] in deriving his equation of motion for non-uniform velocity fields. In Ref. [6], the Faxén corrections are omitted in several terms, because the effect of non-uniformity was simply added in an ad hoc manner to the convective contribution.

Burgers [15] noted that, as a second-order approximation, a small particle "sees" the non-uniform flow around it as a modified flow field given by [16]:

$$\tilde{\mathbf{u}}(\mathbf{z}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t) + \frac{a^2}{6}\nabla^2 \tilde{\mathbf{u}}(\mathbf{x}, t). \tag{57}$$

Analogously, one can argue that a particle moving in a non-uniform temperature field sees the background temperature field as

$$\tilde{T}_{\rm m}(\mathbf{z}, t) = \tilde{T}_{\rm m}(\mathbf{x}, t) + \frac{a^2}{6}\nabla^2 \tilde{T}_{\rm m}(\mathbf{x}, t). \tag{58}$$

Combination of Eqs. (53) and (58) directly yields Eq. (56).

6. Concluding remarks

This work presents several contributions to the understanding of the unsteady heat transfer to small particles in the limit of infinitesimal Pe numbers. First, the governing integro-differential equation is derived in Section 2 by two alternative and previously unexploited methods. Then, a scaling analysis that makes use of the new derivation methods is carried on in Section 3. The analytical solution of the governing equation is compared with the scaling analysis for a few selected cases in Section 4. Finally, the energy equation for a particle moving in non-uniform background field is re-derived and analyzed under the light of the methods described in Section 2.

The derivations in Section 2 are *entirely* performed in physical space, without resorting to transformed-space algebra. In order to accomplish this, we start by making use of Duhamel's Superposition Theorem, and later, we show the origin of the history kernel using Fractional Calculus. To the best of our knowledge, there is no previous publication that relates Duhamel's Theorem explicitly to the history integral nor there is any work using Fractional Calculus that completely dispenses the use of Laplace transformations as has been done here. The novelty in the present derivations is the fact that the governing equations are never integral-transformed. The derivation in physical-space avoids transformed-space algebra, which has been responsible for algebraic mistakes and physical misinterpretations performed by other authors in previous works. A few examples follow:

(a) The solution for the temperature distribution in a slab, Eq. (2) on page 104 of [17], does not satisfy either of the boundary conditions on page 102 of the same book. This inconsistency is due to a misinterpretation of the inverse of the integral transform of the boundary conditions. The difficulty in defining a consistent inversion procedure for all kinds of initial and boundary conditions has been the subject of many studies [18–20]. In our derivation, there is no need for an inversion procedure since the governing equation is solved in physical space.

(b) In the original derivation for the motion of a small particle in a non-uniform velocity field performed by Maxey and Riley [2], the contribution resulting from the integration from $-\infty$ to 0 in the history term was left out of the resulting equation. This discrepancy, which was later corrected (see, e.g. [21]), aroused because Laplace transforms were incorrectly defined in the interval $[0, \infty)$ and not in the interval $(-\infty, \infty)$. Eq. (25) shows the "negative-time" contribution explicitly as a characteristic of the half-derivative operator. There is no need to include extra terms in the operator and the inverse square-root dependence is readily obtained in our formulation.

(c) The kernel of the history term in the present work is often associated with spherically symmetric problems, and it sometimes referred to as "spherically symmetric kernel" (see, e.g. [22]). The form of the kernel of the history term is, however, not related to a particular geometric configuration, but to the fact that diffusion occurs in one dimension only in the coordinate system in question. The kernel of the history term is the same for the unsteady heat diffusion in a planar slab (this follows clearly from Eq. (24)). It is the analog term to the quasi-steady Stokes drag (the second term in the right-hand side of Eq. (21)) that is particular of the spherical symmetry of the problem. This is very clear in our derivation, but is not so clear in the usual Laplace-transform derivation.

The scaling analysis in Section 3 yields the critical frequency above which the history term becomes dominant over the quasi-steady term. The scaling analysis shows that when radiation effects are not relevant, and the scaling number $S = a^2\Omega/\alpha_m$ is of the order of 0.01, the history term contribution accounts for roughly 10% of the total conduction term. For a value of $S \sim 0.1$, the contribution of the history term is approximately 20%; and for $S \sim 1$, the history term contribution is equal to the quasi-steady conduction term contribution. The simplified scaling analysis in Section 3 is accurate as long as phase-differences are not of major relevance for the heat fluxes in question. The accuracy of the scaling analysis, for situations where there is not a significant effect of phase shift between the two quasi-steady and the unsteady heat fluxes, is shown in Fig. 1(a).

Consider, for example, a harmonic temperature field varying in time with an angular frequency Ω. The angular frequency for which history effects become dominant in the heat transfer case is given by $\Omega_c = \alpha_m(1 + C_R)^2/a^2$, where α_m is the thermal diffusivity of the medium, a is the radius of the particles and C_R is the dimensionless radiation coefficient defined in Eq. (38). Unsteady effects become dominant over the quasi-steady diffusion and radiation terms for forcing frequency higher than Ω_c. In combustion environments, the steady-state conduction and radiation effects dominate over the history contribution, unless very high-frequency temperature fluctuations are observed.

As a numerical example of relevance, consider a cloud of particles in air with average diameter of 20 μ. At moderate temperatures ($\tilde{T}_m \approx 500$ K), radiation effects are negligible ($C_R \ll 1$), and $\Omega_c \approx 5 \times 10^5$ rad/s. This value of Ω_c is greater than the oscillating frequencies that are commonly found in fully-turbulent environments. As temperature increases, the value of α_m and C_R increase, thus decreasing the importance of history effects. Larger particles are more affected by the unsteady evolving profile, but the condition of infinitesimal *Pe* number necessary to derive the results obtained in this work are not satisfied for larger particles under usual conditions. However, in micro-gravity combustion environments, convective effects are diminished, thus making possible for history effects to be dominant over quasi-steady ones when particles of millimetric dimensions ($a \geq 1$ mm) are considered.

In liquid–solid suspensions, radiation effects are neg-

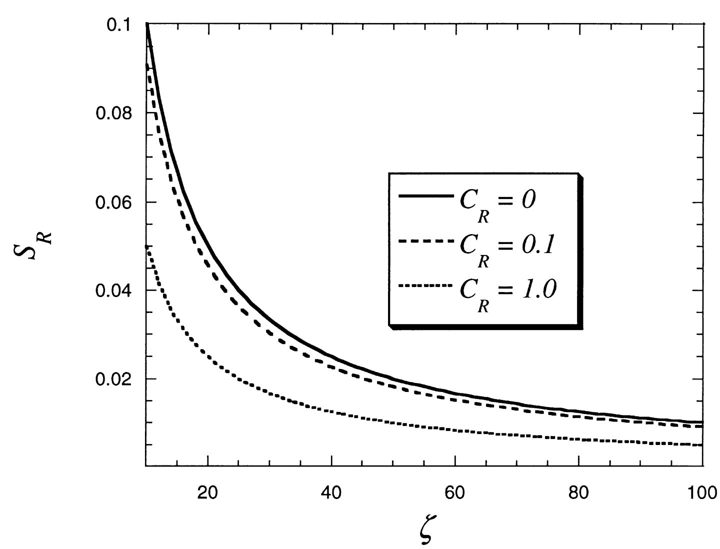


Fig. 3. Approximate behavior of the scaling number S_R with respect to the ratio $\varsigma = \eta_T/a$.

ligible and the effect of the evolving temperature profile around the particles is important even at low frequencies. Consider, for example, a particle with an average diameter of 200 μ suspended in water. For this case, the value of Ω_c is approximately 10 rad/s, a temperature frequency easily attained in turbulent flows. In fact, the Kolmogorov thermal diffusion time scale can be approximated as Ω^{-1}, so that the Kolmogorov thermal length scale η_T is approximately given by α_m/Ω. This implies that $S_R^2 \approx 1/\varsigma^2(1+C_R)$, where $\varsigma = \eta_T/a$. Thus, for a liquid–solid suspension where $C_R \rightarrow 0$, the history contribution increases linearly as ς decreases. However, the assumptions used in the derivation of Eqs. (21) and (22) cannot be satisfied for $\varsigma \leq 10$ because of the non-uniformity of the free-stream condition when $\varsigma \sim a$. The formulation breaks down as $\varsigma \rightarrow a$ and conclusions should not be drawn for these conditions.

For gaseous flows where $Pr = \nu/\alpha_m \rightarrow 1$, the Kolmogorov length scale is of the same order of the Kolmogorov thermal length scale, so that $\eta_K \approx \eta_T$. This implies that in gas–solid flows, the unsteady contribution increases as the Kolmogorov length scale approaches the scale of the radius of the particle, unless radiation heat transfer is the dominant mode. The approximate behavior of the scaling number S_R is depicted in Fig. 3. Note that when ς is of the order of 10–100, the unsteady contribution is still relevant in the absence of strong radiation.

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